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A time-dependent model: bifurcating transition, control and pulsing oscillation

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Abstract
A time-dependent bifurcation model and its control problem are studied. Firstly, delayed bifurcating transition phenomena with memory effects of the model with time-dependent parameters varying in either the positive or the negative direction are analysed. Secondly, a parametric control problem with feedback for the time-dependent model is investigated. The existence and stability of dynamical hysteresis cycles are obtained by qualitative analysis of bifurcation and stability. Finally, an important mechanism for dynamical hysteresis and pulsing oscillation in parametric control systems is revealed as the result of delayed bifurcating transitions when the bifurcation parameter varies periodically across the steady bifurcation value.

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1. Introduction

Time-dependent bifurcations with parameters varying with time exist extensively and have important applications in physics, mechanics, hydrodynamics, chemistry, the life sciences and engineering. There were many such problems in classical non-linear oscillations with parametric excitations. Recently, they have attracted more and more attention since Harberman [1] studied the bifurcation transition and jump phenomena for first- and second-order ordinary differential equations and Erneux and Mandel [2, 3] studied the semi-classical laser equation with a saturable absorber. The phenomena occurring in time-dependent bifurcations are different from those in steady bifurcations, for example, there may exist bifurcating transitions, jumps, delay and memory effects and dynamical bistability [4–9]. Some of these phenomena are permitted, for instance, in bistable systems in laser physics [4–6]; but some are harmful, for instance, in chemistry [7], mechanics and engineering [8]. In addition, the control problem for time-dependent bifurcations is also proposed and has not been researched extensively to date.

Time-delay phenomena and pulsing oscillations occur in many control and communication problems, such as laser pulses [3], the Brusselator in catalytic reactions, singular Hopf bifurcations and relaxation oscillations in nerve transport activities [9]. Usually, these systems are non-linear, non-autonomous and difficult to treat even in the cases with slowly varying parameters. Relaxation oscillations and time-dependent bifurcations in systems with slowly varying parameters have been investigated by methods of singular perturbation, qualitative analysis and numerical simulation in the literature (e.g., [10–12]). However, their general mechanisms are not wholly clear thus far.

In this paper, first a time-dependent bifurcation model is considered, and delayed bifurcating transition phenomena with memory effects are analysed when the bifurcation parameter varies with time in a general way. Next, a control problem with time-dependent parametric feedback for the model is studied. A dynamical hysteresis cycle and its stability can be found through the qualitative analysis of Hopf–Poincaré bifurcation and the calculation of the index number. Finally, an important mechanism of pulsing oscillations due to delayed bifurcating transitions in parametric control systems is revealed.

2. A time-dependent bifurcation model

Consider the following time-dependent bifurcation model:

$$\dot{y} = \lambda(t)(y - y^{n+1}) \tag{1}$$

where $\lambda(t)$ is a time-dependent bifurcation parameter and n is a positive integer. There is no need to assume that $\lambda(t)$ is a slowly varying parameter in this paper. The steady bifurcation problem corresponding to equation (1) is given as follows:

$$\dot{y} = \lambda(y - y^{n+1}) \tag{2}$$

where the steady bifurcation parameter λ is a constant.

Equation (1) or (2) can be used to investigate dynamical mechanisms of some non-linear oscillators, such as relaxation oscillations of electronic circuits, jump processes of the heart, on-off systems with memory effects and time-dependent models of insect population. Equation (1) can also be considered as an active parametric control system with a given control variable $\lambda(t)$. The dynamical system given by equation (2) with a linear feedback control

$$\begin{cases} \dot{y} = \lambda(y - y^{n+1}) + z \\ \dot{z} = -y \end{cases} \tag{3}$$

is equivalent to a generalized van der Pol-type non-linear oscillator $\ddot{y} + \lambda[(n+1)y^n - 1]\dot{y} + y = 0$. The dynamical behaviour of system (1) with a general time-dependence of $\lambda(t)$ and its parametric control problem have not yet been studied.

The steady bifurcation analysis of equation (2) is given briefly first of all. Let $F(\lambda, y) = \lambda(y - y^{n+1})$. From $F(\lambda, y) = 0$, we know that the equilibrium solutions of equation (2) are determined by $y = 0$ and $y^n = 1$ when $\lambda \neq 0$, and their stabilities can be determined by the derivative $F_y(\lambda, y) = \lambda(1 - (n+1)y^n)$. The whole y-axis contains non-hyperbolic equilibrium solutions when $\lambda = 0$. For example, when n is an even number, equation (2) has three equilibrium solutions $y = 0$ and $y = \pm 1$ for $\lambda \neq 0$. When $\lambda < 0$, $F_y(\lambda, 0) = \lambda < 0$ and $F_y(\lambda, \pm 1) = -n\lambda > 0$, respectively; hence $y = 0$ is stable and $y = \pm 1$ are both unstable. When $\lambda > 0$, $F_y(\lambda, 0) = \lambda > 0$ and $F_y(\lambda, \pm 1) = -n\lambda < 0$, respectively; hence $y = 0$ is unstable and $y = \pm 1$ are both stable. Therefore, the steady bifurcations of equation (2) can be considered as either subcritical or supercritical according to the bifurcation parameter λ varying in the positive or negative direction, and the steady bifurcation

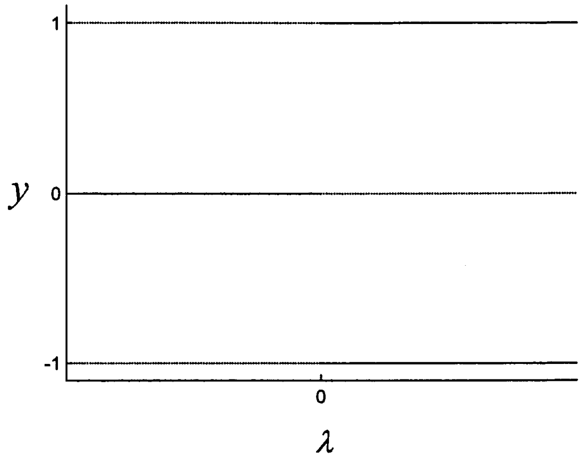


Figure 1. The steady bifurcation diagram of equation (2) when n is an even number.

value is $\lambda_c = 0$. Figure 1 shows the steady bifurcation diagram of equation (2) when n is an even number. Although the number of equilibrium solutions does not vary before and after the bifurcation occurs, we see that the stability of the equilibrium solutions exchanges suddenly. Actually, the steady bifurcation of equation (2), which differs from well-known bifurcations in the literature, is of higher co-dimension and degeneracy. The steady bifurcation of equation (2) when n is an odd number can be discussed similarly and is omitted here.

Now turn to the time-dependent bifurcation system given by equation (1), which is a non-linear, non-autonomous system having some different dynamic properties with the corresponding autonomous one given by equation (2), especially the delayed bifurcating transition phenomena with memory effects. Because, in general, the bifurcation parameter $\lambda(t)$ in equation (1) need not be slowly varying, the usual asymptotic methods or scale balancing are not applicable. Here, the time-dependent bifurcation problem of equation (1) can be analysed by means of the integrability in a simpler way.

In order to investigate the relation between the solutions of equations (1) and (2) and the effect of the time-dependence of $\lambda(t)$, it is necessary to introduce the following concepts about bifurcating transitions. Suppose that there is an equilibrium solution $y = y_{s1}$ of equation (2), which is stable for $\lambda < \lambda_c$. Moreover, another equilibrium solution $y = y_{s2}$ of equation (2) bifurcates from y_{s1} at $\lambda = \lambda_c$ and is stable for $\lambda > \lambda_c$. Let $y(t)$ be a solution of equation (1). If there exists a small quantity $\delta > 0$, and $t_{nc}, t_{nf} (t_{nf} \geqslant t_{nc})$ such that $|y(t) - y_{s1}| < \delta$ for $t < t_{nc}$ and $|y(t) - y_{s2}| < \delta$ for $t > t_{nf}$, then it is seen that the solution $y(t)$ undergoes a bifurcating transition within δ-extent from y_{s1} to y_{s2} in the time interval (t_{nc}, t_{nf}). Denote $\lambda_{nc} = \lambda(t_{nc})$ and $\lambda_{nf} = \lambda(t_{nf})$; t_{nc} and $\lambda_{nc} = \lambda(t_{nc})$ are called the transition time and the transition value, respectively; (t_{nc}, t_{nf}) (or $(\lambda_{nc}, \lambda_{nf})$) is called the transition time (or parameter) interval. Usually, t_{nc}, t_{nf} and $\lambda_{nc}, \lambda_{nf}$ are dependent on δ. If $|t_{nf} - t_{nc}| \to 0$ (or equivalently, $|\lambda_{nf} - \lambda_{nc}| \to 0$) as $\delta \to 0$, the bifurcating transition is called a jump; that is, $y(t)$ varies rapidly from y_{s1} to y_{s2} when δ tends to zero. The bifurcating transition has a memory effect if the transition direction is determined by the initial value of $y(t)$.

Let $\lambda(t)$ vary in the positive direction, that is, $\lambda(t)$ increases monotonically with respect to t. The bifurcating transition from y_{s1} to y_{s2} is delayed if $\lambda_{nc} > \lambda_c$ and is advanced if $\lambda_{nc} < \lambda_c$. Conversely, let $\lambda(t)$ vary in the negative direction, that is, $\lambda(t)$ decreases monotonically with respect to t. The bifurcating transition from y_{s2} to y_{s1} is delayed if $\lambda_{nc} < \lambda_c$ and is advanced if $\lambda_{nc} > \lambda_c$.

In the following proposition, only the cases of even numbers n are considered. The cases of odd numbers n can be discussed similarly.

Proposition 1. *Let n be an even number and $y(t)$ be a solution of equation (1) with $0 < y(0) = y_0 < 1$ (or $-1 < y_0 < 0$). Denote $\delta = |y_0|$. If the bifurcation parameter $\lambda(t)$ varies in the positive direction with $\lambda(0) < 0$ and δ is small enough, then $y(t)$ undergoes a bifurcating transition within δ-extent from $y_{s1} = 0$ to $y_{s2} = 1$ (or -1) in a transition interval (t_{nc}, t_{nf}), where t_{nc} is given by*

$$\int_0^{t_{nc}} \lambda(s)\,\mathrm{d}s = 0 \tag{4}$$

and t_{nf} is given by

$$\int_0^{t_{nf}} \lambda(s)\,\mathrm{d}s = -\left(1 + \frac{1}{n}\right)\ln\delta. \tag{5}$$

Moreover, the transition value

$$\lambda_{nc} = \lambda(t_{nc}) > \lambda_c = 0. \tag{6}$$

Similar results, except that $\lambda_{nc} < \lambda_c = 0$, can be obtained for the bifurcating transition from $y_{s2} = 1$ (or -1) to $y_{s1} = 0$ if $\lambda(t)$ varies in the negative direction with $\lambda(0) > 0$. Therefore, the bifurcating transitions in both directions are delayed and have memory effects.

Proof. Consider the case of an even number n and $0 < y_0 = \delta < 1$ in detail. Let the equilibrium solutions of equation (2) be $y = y_{s1} = 0$ and $y = y_{s2} = 1$ with the stability and steady bifurcations discussed above. Integrating equation (1) from 0 to t yields

$$\frac{y(t)}{\delta}\left[\frac{\delta^n - 1}{y^n(t) - 1}\right]^{\frac{1}{n}} = \exp\left(\int_0^t \lambda(s)\,\mathrm{d}s\right). \tag{7}$$

It follows from equation (7) that $0 < y(t) < 1$ for $t \in (0, +\infty)$ and $y(t) \to 1$ when $t \to +\infty$. Now the dynamical behaviour of $y(t)$ for $t \in (0, +\infty)$ is considered.

It is obvious from equation (7) that $y(t) = \delta$ iff t satisfies

$$\int_0^t \lambda(s)\,\mathrm{d}s = 0. \tag{8}$$

Now, if $\lambda(t)$ varies in the positive direction with $\lambda(0) < 0$, then there is a unique t_{nc} satisfying equation (8) such that $y(t_{nc}) = \delta$. Hence, t_{nc} is determined by equation (4). Since $\int_0^t \lambda(s)\,\mathrm{d}s < 0$ for $t \in (0, t_{nc})$, equation (7) leads to

$$0 < |y(t)| < \delta \qquad \text{for} \quad t \in (0, t_{nc}). \tag{9}$$

Furthermore, it can be shown that $\lambda(t_{nc}) > 0$ by equation (4) since $\lambda(0) < 0$ by assumption, and then there is $\bar{t} \in (0, t_{nc})$ such that $\lambda(\bar{t}) = 0$. Hence, noting that $0 < y^n(t) < y(t) < 1$ since $0 < y(t) < 1$ for $t \in (0, +\infty)$, it is shown from equation (1) that $\dot{y} < 0$ for $t \in (0, \bar{t})$ and $\dot{y} > 0$ for $t \in (\bar{t}, t_{nc})$; that is, $y(t)$ decreases in $(0, \bar{t})$ and increases in $(\bar{t}, t_{nc})$.

Again, it will be shown that

$$0 < |y(t) - 1| < \delta \qquad \text{for} \quad t \in (t_{nf}, +\infty) \tag{10}$$

where t_{nf} is given by equation (5). In fact, on the one hand, it follows from equation (5) that t_{nf} satisfies

$$\exp\left(\int_0^{t_{nf}} \lambda(s)\,\mathrm{d}s\right) = \delta^{-(1+\frac{1}{n})}. \tag{11}$$

Since $\lambda(t)$ increases monotonically with respect to t, $\int_0^{t_{nf}} \lambda(s)\,\mathrm{d}s < \int_0^{t} \lambda(s)\,\mathrm{d}s$ for $t > t_{nf}$. Then equation (11) leads to

$$\begin{aligned}\delta^{(1+n)} &= \exp\left(-n\int_0^{t_{nf}} \lambda(s)\,\mathrm{d}s\right)\\ &> \exp\left(-n\int_0^{t} \lambda(s)\,\mathrm{d}s\right)\end{aligned} \tag{12}$$

for $t \in (t_{nf}, +\infty)$. On the other hand, it follows from $0 < y(t) < 1, 0 < \delta < 1$ and equation (7) that

$$\begin{aligned}|y^n(t) - 1| &= \frac{y^n(t)|\delta^n - 1|}{\delta^n}\exp\left(-n\int_0^{t} \lambda(s)\,\mathrm{d}s\right)\\ &< \delta^{-n}\exp\left(-n\int_0^{t} \lambda(s)\,\mathrm{d}s\right).\end{aligned} \tag{13}$$

Since $0 < y(t) < 1$, from inequalities (12) and (13) it is clear that

$$|y(t) - 1| < |y^n(t) - 1| < \delta^{-n}\delta^{(1+n)} = \delta$$

for $t \in (t_{nf}, +\infty)$ and then inequality (10) is proved.

Let δ be small enough so that $t_{nf} \geqslant t_{nc}$. It is concluded from inequalities (9) and (10) that in this case there is a bifurcating transition of $y(t)$ within δ-extent from $y_{s1} = 0$ to $y_{s2} = 1$ in the time interval (t_{nc}, t_{nf}).

A similar conclusion can be made for the case of $-1 < y_0 = -\delta < 0$ with small $\delta > 0$ and $\lambda(t)$ varying in the positive direction with $\lambda(0) < 0$. In this case, $-1 < y(t) < 0$ for $t \in (0, +\infty)$ and $y(t) \to -1$ when $t \to +\infty$. Inequalities (9) and (10) still hold, and then there is a bifurcating transition of $y(t)$ within δ-extent from $y_{s1} = 0$ to $y_{s2} = -1$ in the time interval (t_{nc}, t_{nf}).

In summary, recall that in the above discussion the steady bifurcation value of equation (2) is $\lambda_c = 0$ and the bifurcating transition value of equation (1) is $\lambda_{nc} = \lambda(t_{nc}) > 0$. Then it is clear that $\lambda(t_{nc}) > \lambda_c$ and the bifurcating transition of $y(t)$ is delayed. The transition direction of $y(t)$ from $y_{s1} = 0$ to $y_{s2} = 1$ (or -1) depends on the initial value $y(0) > 0$ (or < 0), so the transition has a memory effect.

Conversely, if $\lambda(t)$ varies in the negative direction with $\lambda(0) > 0$, then similar conclusions can be made for the bifurcating transition of $y(t)$ from $y_{s2} = 1$ (or -1) to $y_{s1} = 0$, except that $\lambda_{nc} < \lambda_c = 0$. Therefore, the bifurcating transition is still delayed and has a memory effect. This completes the proof. □

The discussion is similar for the cases of odd numbers n.

As an important example, consider the case of $\lambda(t)$ varying linearly with respect to t; that is, $\lambda(t) = \alpha t + \lambda_0$, where α and λ_0 are constants. If $\alpha > 0$ and $\lambda_0 < 0$, then $\lambda(t)$ varies in the positive direction with $\lambda(0) = \lambda_0 < 0$. By equations (9) and (10) it is easy to get

$$t_{nc} = -\frac{2\lambda_0}{\alpha} \qquad t_{nf} = \frac{1}{\alpha}\left[-\lambda_0 + \sqrt{\lambda_0^2 - 2\alpha\left(1 + \frac{1}{n}\right)\ln\delta}\right]. \tag{14}$$

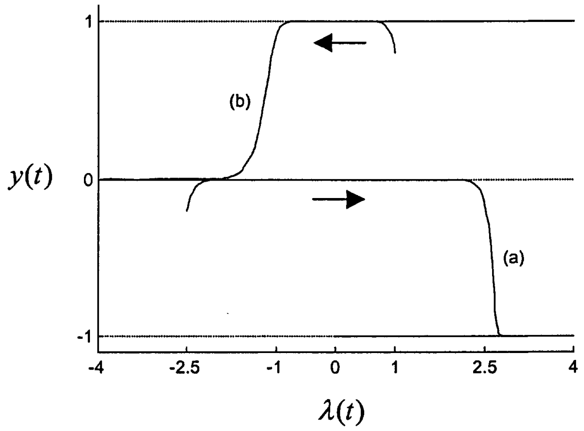


Figure 2. Bifurcating transitions of equation (1) in two directions for $n = 4$.

Obviously,

$$\lambda_{nc} = \lambda(t_{nc}) = -\lambda_0 \qquad \lambda_{nf} = \lambda(t_{nf}) = \sqrt{\lambda_0^2 - 2\alpha\left(1 + \frac{1}{n}\right)\ln\delta}. \tag{15}$$

It is seen from equation (15) that $\lambda_{nc} > \lambda_c = 0$ since $\lambda_0 < 0$, and then the bifurcating transition of $y(t)$ from $y_{s1} = 0$ to $y_{s2} = 1$ (or -1) is delayed. Conversely, if $\alpha < 0$ and $\lambda_0 > 0$, then $\lambda(t)$ varies in the negative direction with $\lambda(0) = \lambda_0 > 0$. In this case, t_{nc}, t_{nf} and λ_{nc}, λ_{nf} are still given by equations (14) and (15), respectively. It is seen that $\lambda(t_{nc}) < \lambda_c$ since $\lambda_0 > 0$, and then the bifurcating transition of $y(t)$ from $y_{s2} = 1$ (or -1) to $y_{s1} = 0$ is still delayed. The bifurcating transitions in both the directions have memory effects from the general discussion in proposition 1; however, they are not jumps since t_{nf} tends to infinity as δ tends to zero by equation (14).

Figure 2 shows the bifurcating transitions of equation (1) for $n = 4$. Curve (a) denotes a bifurcating transition in the positive direction from $y = 0$ to $y = -1$, where the time-dependent bifurcation parameter $\lambda(t) = -2.5 + 0.25t$ and the initial value $y_0 = -0.25$. Obviously, $\lambda_0 = -2.5, \alpha = 0.25$ and $\delta = 0.25$. From equation (15) it is found that $\lambda_{nc} = 2.5 > \lambda_c = 0$ and $\lambda_{nf} = 2.67$. It is seen that there is a bifurcating transition within δ-extent in the parameter interval (2.5, 2.67). Curve (b) denotes a bifurcating transition in the negative direction from $y = 1$ to $y = 0$, where $\lambda(t) = 1 - 0.25t$ and $\lambda_{nc} = -1 < \lambda_c = 0$. It is seen from figure 2 that the bifurcating transitions in both directions are delayed and have memory effects. The corresponding steady bifurcation diagram is also plotted in figure 2 for comparison.

3. Control of the time-dependent bifurcation model

In the above section, it is seen that the bifurcating transitions of equation (1) for the parameter $\lambda(t)$ varying in both the positive and negative directions are delayed and have memory effects. Now, if $\lambda(t)$ changes its directions periodically, it can be imagined that the delayed bifurcating transitions of equation (1) cause a hysteresis cycle, which exhibits a kind of relaxation oscillation with pulsing behaviour.

In this section, a parametric control problem for the time-dependent bifurcation system (1) is discussed. Considering a linear feedback control for equation (1) illustrated in figure 3 and

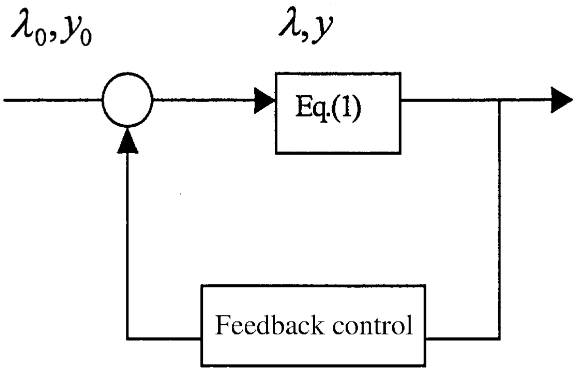


Figure 3. Illustration of feedback control for equation (1).

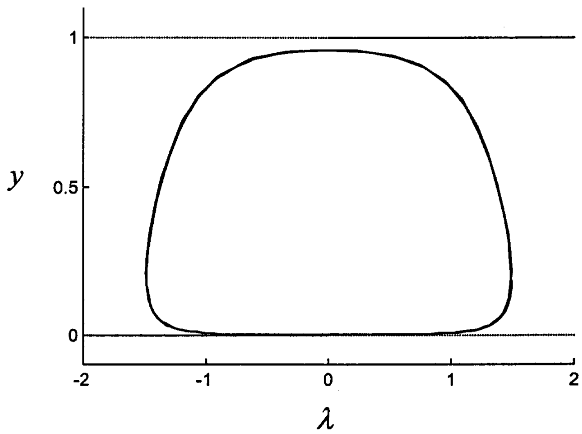


Figure 4. Dynamical hysteresis cycle of equation (5) ($n = 3$).

taking $\lambda(t)$ as the control variable, the system is described by the following equations:

$$\begin{cases} \dot{\lambda} = a(b + G\lambda - cy) \equiv aE(\lambda, y) \\ \dot{y} = \lambda(y - y^{n+1}) \equiv F(\lambda, y) \end{cases} \tag{16}$$

where $n > 0$ is an integer, a and b are positive constants, $c > 0$ is a feedback coefficient and G is a bifurcation parameter. When $G > 0$, it will be shown later that the feedback control makes the parameter $\lambda(t)$ vary in both the positive and negative directions periodically and yields a periodic solution of equations (16), which corresponds to a dynamical hysteresis cycle as depicted in figure 4. In order to discuss the generation of the periodic solution, it is necessary to deal with the Hopf–Poincaré bifurcation of equations (16).

When $G = 0$, equations (16) are in fact an integrable system as follows:

$$\begin{cases} \dot{\lambda} = a(b - cy) \\ \dot{y} = \lambda(y - y^{n+1}). \end{cases} \tag{17}$$

It has a first integral given by $H(\lambda, y) = \frac{1}{2}\lambda^2 + \int \frac{a(cy-b)}{y-y^{n+1}}\,\mathrm{d}y = \text{const}$. The unique equilibrium point $(0, b/c)$ is a centre. All the other orbits are closed cycles around the centre.

When $G \neq 0$, equations (16) have three equilibrium points, namely $(-b/G, 0)$, $((c-b)/G, 1)$ and $(0, b/c)$. In addition, assume that $b < c$ in the following discussion. There is no closed cycle around the first two equilibrium points because they are saddle points. As for the third equilibrium point $(0, b/c)$, the Jacobian matrix has a pair of complex conjugate eigenvalues given by $r_{1,2} = [aG \pm \sqrt{(aG)^2 - 4ab(1-(b/c)^n)}]/2$ when $|G| < 2\sqrt{b(1-(b/c)^n)/a}$. The real and imaginary parts are $\alpha(G) = aG/2$ and $\beta(G) = \sqrt{4ab(1-(b/c)^n) - (aG)^2}/2$, respectively. The equilibrium point $(0, b/c)$ is a stable focus when $-2\sqrt{b(1-(b/c)^n)/a} < G < 0$ and is an unstable focus when $0 < G < 2\sqrt{b(1-(b/c)^n)/a}$, but it is a centre when $G = 0$. The eigenvalues $r_{1,2}$ become a pair of imaginary eigenvalues when $\alpha(G) = 0$, that is, $G = 0$. Since $\alpha(0) = 0$ and $\alpha'(0) = a/2 > 0, G = G_h = 0$ is a Hopf–Poincaré bifurcation value of equations (16) according to the Hopf–Poincaré bifurcation theorem, and then it can be shown that for $G > 0$ small enough and $c > b$ there exists a stable closed cycle of equations (16) around the equilibrium point $(0, b/c)$; in other words, under this situation $\lambda(t)$ and $y(t)$ vary periodically with respect to t. The period of the closed cycle is $T \approx 2\pi/\beta(0)$ for sufficiently small values of $G > 0$.

Furthermore, since the stability of the closed cycle can also be determined by calculating its index number, this leads to the following result:

Proposition 2. *Let* $0 < G < 2\sqrt{b(1-(b/c)^n)/a}$, $b < c$ *and* $(\lambda(t), y(t))$ *be a closed cycle of equations (16) with period T around the equilibrium point* $(0, b/c)$. *Denote the averaged value of* $y(t)$ *and* $\lambda(t)$ *by* $\overline{y} = \frac{1}{T}\int_0^T y(t)\,\mathrm{d}t$ *and* $\overline{\lambda} = \frac{1}{T}\int_0^T \lambda(t)\,\mathrm{d}t$, *respectively. Then the closed cycle is stable when* $aG^2 + nb < nc\overline{y}$ *(or equivalently,* $G < n\overline{\lambda}/a$), *and it grows into a large dynamical hysteresis cycle as the parameter G increases.*

Proof. Let the initial value $y(0) = y_0$ of the closed cycle be in (0, 1). Because the second equation of (16) is just the same as equation (1), it is known from equation (7) that the closed cycle of equations (16) exists in the strip-shape domain $0 < y < 1$.

Integrating the first equation of (16) along the closed cycle in one period gives

$$\overline{\lambda} = -\frac{b}{G} + \frac{c}{G}\overline{y}. \tag{18}$$

Dividing both sides of the second equation of (16) by $y(t)$ and then integrating along the closed cycle in one period, it follows that

$$\overline{\lambda} = \frac{1}{T}\int_0^T \lambda(t) y^n(t)\,\mathrm{d}t. \tag{19}$$

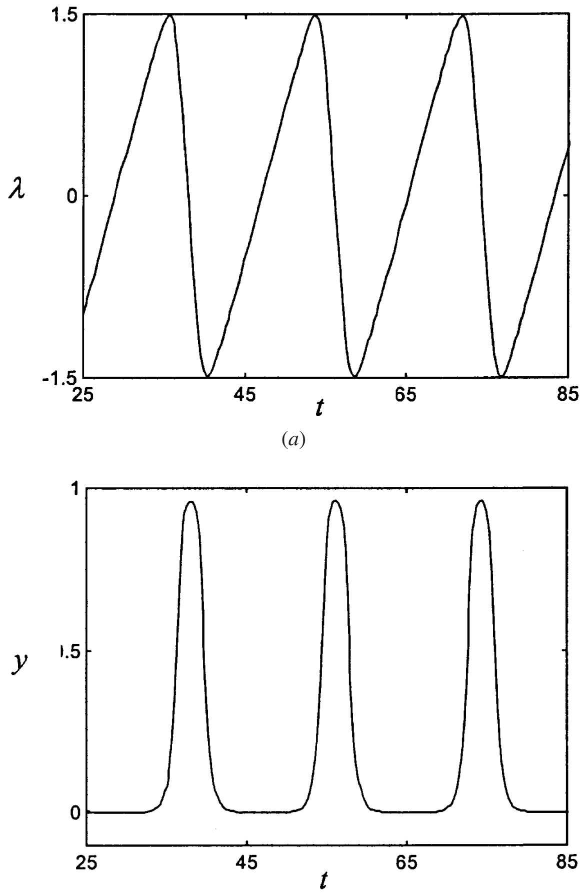
Thus the index number γ of the closed cycle can be computed by equations (16) and (19) as follows:

$$\begin{aligned}\gamma &= \frac{1}{T}\int_0^T \left(a\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial y}\right)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T [aG + \lambda(t) - (n+1)\lambda(t)y^n(t)]\,\mathrm{d}t \\ &= aG - n\overline{\lambda}.\end{aligned} \tag{20}$$

Moreover, substituting equation (18) into equation (20) gives

$$\gamma = (aG^2 + nb - nc\overline{y})/G. \tag{21}$$

Therefore, if $aG^2 + nb < nc\overline{y}$ (or equivalently, $G < n\overline{\lambda}/a$) holds for appropriately chosen values of a, b, c, n and $G > 0$, then the index $\gamma < 0$ from equation (21) (or equation (20))



(*a*)

(*b*)

Figure 5. Periodic variation of $\lambda(t)$ and pulsing oscillation of $y(t)$ of equations (16).

and the closed cycle is stable. Especially, since the closed cycle is in the strip-shape domain $0 < y < 1$, it follows that $0 < \overline{y} < 1$ and then the feedback parameter G for a stable closed cycle must satisfy $0 < G < G_{cr} = \sqrt{n(c-b)/a}$.

Furthermore, as the parameter G increases, the stable limit cycle grows larger and larger with its amplitude increasing. Because $\mathrm{d}y/\mathrm{d}\lambda = \lambda(y - y^{n+1})/[a(b + \lambda G - cy)] \to 0$ as $y(t) \to 0^+$ (or $y(t) \to 1$) from equations (16), it is seen that for sufficiently large $G > 0$ the closed cycle is so large that it is almost tangential to the line $y = 0$ and the line $y = 1$ simultaneously. Taking account of the delayed bifurcating transitions between $y = 0$ and $y = 1$, it follows that the closed cycle becomes a dynamical hysteresis cycle as G increases (see figure 4), where $y(t)$ varies rapidly between $y = 0$ and $y = 1$ in the transition intervals and exhibits pulsing behaviour. This completes the proof. □

Numerical results in figure 5 show the pulsing oscillation of $y(t)$ along with the periodic variation of the control variable $\lambda(t)$, which correspond to the dynamical hysteresis cycle of equations (16) plotted in figure 4. In the computation, $n = 3$, $a = 0.025$, $b = 10$, $c = 50$ and $G = 0.005$. Actually, this means that pulsing oscillations may occur in the feedback control system (16), even for rather small values of the feedback coefficient G.

4. Conclusion

Combining the above results, it is seen that time-dependent bifurcations play an important rôle in time-dependent systems. There exist extensive dynamical phenomena, especially bifurcating transitions with delay and memory effects. In general, for time-dependent control systems, there may exist dynamical hysteresis cycles due to the delayed bifurcating transitions induced by the time-dependent control parameters passing through the steady bifurcation values periodically, and this leads to the pulsing behaviour of oscillations in the systems. Hence, an important mechanism for the generation of time-delays and pulsing oscillations in parametric feedback control systems is revealed.

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